Subject	Class	Paper	Study materials on	Resource Person	
Mathematics	D-III (H)	V	Complex Analysis	Dr. S. Ahmed	Associate Professor

Complex number

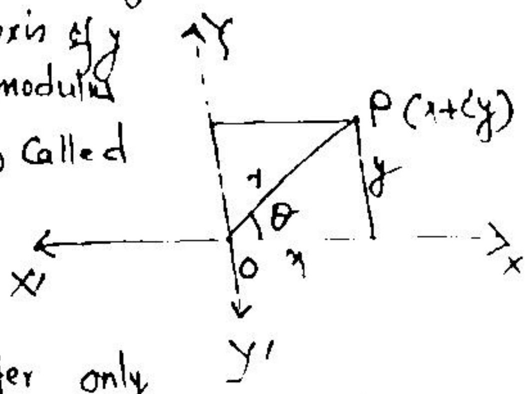
A number of the form $x+iy$ is called a complex number where x and y are both real numbers and $i=\sqrt{-1}$. Here x is called real part & y is called imaginary part of the complex number $x+iy$. A complex number $x+iy$ is denoted by z.

$$\text{Thus } z = x+iy$$
$$x = \text{Real part of } z = Re(z)$$
$$y = \text{imaginary part of } z = Im(z)$$

The complex number $z=x+iy$ is called purely real if $y=0$ and is purely imaginary if $x=0$

Argand diagram

Mathematician Argand represented a complex number in a diagram known as Argand diagram. A complex number $x+iy$ can be represented by a point P whose co-ordinates are (x,y). The axis of x is called the real axis and the axis of y the imaginary axis. The distance OP is the modulus and the angle θ, OP makes with x-axis, is called the arguement of $x+iy$.



Conjugate of a Complex number

Two complex numbers which differ only in the sign of imaginary parts are called conjugate of each other.

A pair of complex numbers $x+iy$ and $x-iy$ are said to be conjugate of each other.

The conjugate of $z=x+iy$ is denoted by $\bar{z} = x-iy$

Modulus and Arguement

Let $z = x+iy$ be a complex number

Putting $x = r\cos\theta$ & $y = r\sin\theta$ so that $r=\sqrt{x^2+y^2}$

$$\therefore \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}} \quad \& \quad \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

Then r is called the modulus or absolute value of the Complex number $x+iy$ and is denoted by $|z| = |x+iy|$.

The angle θ is called the arguement or amplitude of the Complex number $x+iy$ and denoted by $Arg\, z = Arg\,(x+iy)$

It is clear that θ will have infinite number of values differing by multiples of 2π. The values of θ lying in the range $-\pi < \theta \leq \pi$ $[(0<\theta<\pi)$ or $(-\pi < \theta < 0)]$ is called the Principal value of the arguement.

The principal value of θ is written either between $0 \,\&\, \pi$ or between $0 \,\&\, -\pi$.

The Triangular inequality

(a) The modulus of sum and difference of two Complex numbers is always less than or equal to sum of their moduli.

i.e. to prove that $|z_1+z_2| \leq |z_1|+|z_2|$

and $|z_1-z_2| \leq |z_1|+|z_2|$

Proof Let z_1 and z_2 be two Complex numbers.

Then $|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$ $\quad [\because z\bar{z} = |z|^2]$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 + |z_2|^2$

$= |z_1|^2 + |z_2|^2 + (z_1\bar{z}_2 + z_2\bar{z}_1)$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$: Since $Re\, z < |z|$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2|$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \quad \left(\because |\bar{z}_2| = |z_2|\right)$$

$$= (|z_1| + |z_2|)^2$$

Hence $|z_1+z_2| \leq |z_1|+|z_2|$

Writing $-z_2$ for z_2 in this result, We have

$$|z_1-z_2| \leq |z_1| + |-z_2|$$

$$\Rightarrow |z_1-z_2| \leq |z_1| + |-z_2| \text{ i.e. } |z_1-z_2| \leq |z_1| + |z_2|$$

$[\because |-z_2| = |z_2|]$ Proved

ⓑ The modulus of difference or sum of two complex numbers is always greater than or equal to diffence of their moduli.

<u>Proof:</u> We have $|z_1-z_2|^2 = (z_1-z_2)\overline{(z_1-z_2)}$

$$= (z_1-z_2)(\bar{z}_1-\bar{z}_2)$$

$$= z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 - (z_1\bar{z}_2 + z_2\bar{z}_1) + |z_2|^2$$

$$= |z_1|^2 + |z_2|^2 - 2Re(z_1\bar{z}_2)$$

$$\geq |z_1|^2 + |z_2|^2 - 2|z_1\bar{z}_2|$$

Since $Re(z_1\bar{z}_2) \leq |z_1\bar{z}_2|$

$$= |z_1|^2 + |z_2|^2 - 2|z_1||\bar{z}_2|$$

$$= |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$$= (|z_1| - |z_2|)^2$$

Hence $|z_1-z_2| \geq |z_1| - |z_2|$

Writing $-z_2$ for z_2, we have

$|z_1+z_2| \geq |z_1| - |-z_2|$ i.e. $|z_1+z_2| \geq |z_1| - |z_2|$

Hence the result.